DYONS IN NONABELIAN BORN–INFELD THEORY

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Abstract
We analyze the nonabelian extension of Born–Infeld lagrangian for $SU(2)$ group. In the class of spherically symmetric solutions of finite energy, besides the Gal'tsov–Kerner glueballs we find only the analytic dyon solutions.

1. Action and field equations

The initial point of our analysis is the following nonabelian Born–Infeld [1, 2] (NBI) action in Minkowski space:

$$S = \frac{1}{4\pi}\int d^4x(1-\mathcal{R}) \ , \ \mathcal{R} = \sqrt{1+\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{16}F^a_{\mu\nu}F^{*\mu\nu a}} \ . \qquad (1)$$

The equations of motion which follow from the NBI action (1) are

$$D_\mu \frac{F^{\mu\nu} - GF^{*\mu\nu}}{\mathcal{R}} = 0 \ . \qquad (2)$$

Here F^* denotes the Hodge-dual, $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, D_μ is covariant derivative, a is the index of the gauge group and $G = \frac{1}{4}F^a_{\mu\nu}F^{*\mu\nu a}$. The equations of motion (2) can be complemented with the Bianchi identities, $D_\mu F^{*\mu\nu} = 0$.

In [2], spherically-symmetric configurations of finite energy for the action (1) were found. The ansatz for the gauge potentials was the monopole ansatz and it describes purely magnetic configurations. The usual splitting of the field strengths $F^a_{\mu\nu}$ into "electric" and "magnetic" parts is:

$$E^a_i = F^a_{i0} \ , \quad B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk} \ . \qquad (3)$$

We will generalize the monopole ansatz – in fact, we will consider the general spherically symmetric static potential for the $SU(2)$ group (the so-called Witten's ansatz, [3]). It is given by three real functions $a_0(r)$, $a_1(r)$, $w(r)$ of the radial coordinate r. The components of the gauge potential read:

$$A^a_0 = a_0(r)\frac{x^a}{r} \ , \ A^a_i = a_1(r)\frac{x^a x^i}{r^2} + \epsilon_{aik}\frac{1-w(r)}{r}\frac{x^k}{r} \ . \qquad (4)$$

Here x^a, x^i and x^k are the Cartesian coordinates. The field strengths for this ansatz are

$$E_i^a = a_0' \,\frac{x_i x_a}{r^2} - \frac{a_0 w}{r}\,\frac{x_i x_a - \delta_{ia} r^2}{r^2} \,,$$

$$B_i^a = -2\delta_{ia}\,\frac{1-w}{r^2} + \frac{(1-w)^2}{r^2}\,\frac{x_i x_a}{r^2} + \Big(\frac{1-w}{r^2}\Big)'\,\frac{x_i x_a - \delta_{ia} r^2}{r} + \frac{a_1 w}{r^2}\,\epsilon_{iak} x_k \,,$$

where prime denotes the derivative $\frac{d}{dr}$. The square root $\mathcal{R}$ from (1) is

$$\mathcal{R} = \sqrt{1 + \frac{(1-w^2)^2}{r^4} + 2\frac{w'^2}{r^2} + 2\frac{a_1^2 w^2}{r^2} - 2\frac{a_0^2 w^2}{r^2} - a_0'^{\,2} - \frac{[a_0(1-w^2)]'^2}{r^4}} \,.$$

We can now consider the condition of extremality of the action. After the integration of angular variables, the action is proportional to the lagrangian L,

$$L = \int_0^\infty r^2(\mathcal{R}-1)dr \,. \tag{5}$$

Varying the unknown functions a_0, a_1 and w, we obtain the set of equations:

$$w^2 a_1 = 0 \,, \tag{6}$$

$$(1-w^2)\left(\frac{[a_0(1-w^2)]'}{r^2\mathcal{R}}\right)' = \frac{2w^2 a_0}{\mathcal{R}} - \left(\frac{r^2 a_0'}{\mathcal{R}}\right)' , \tag{7}$$

$$w a_0 \left(\frac{[a_0(1-w^2)]'}{r^2\mathcal{R}}\right)' = -\frac{2w(1-w^2)}{r^2\mathcal{R}} - \left(\frac{2w'}{\mathcal{R}}\right)' - \frac{w a_0^2}{\mathcal{R}} + \frac{w a_1^2}{\mathcal{R}} \,. \tag{8}$$

2. NBI dyons

The system of equations (6–8) is a complicated nonlinear system. We will search for particular solutions of this system with finite energy. The energy of the static configurations is equal to the negative value of the lagrangian, $M = -L$. The convergence of this integral on both boundaries imposes restrictions on the asymptotic behavior of the functions a_0, a_1 and w. We will discuss these restrictions later.

The simplest equation (6) implies essentially that $a_1(r) = 0$. Therefore, we will always assume this and denote $a_0(r) = a(r)$ in the following.

The solutions with $a(r) = 0$ were analyzed in [2] in detail. The simplest solution in this case is $w(r) = \pm 1$ and it represents the pure gauge. $w(r) = 0$ is also a solution: this is the embedded $U(1)$ monopole. Its energy is finite:

$$M_e = \frac{\pi^{3/2}}{3\Gamma(3/4)^2} \approx 1.2360 \,. \tag{9}$$

There is also an infinite discrete set of finite energy solutions $w_n(r)$ with the index $n \in \mathbb{N}$. These solutions can be found numerically using the condition that the function $w(r)$ with the allowed asymptotic forms at $r \to 0$ and $r \to \infty$ match in the intermediate region. They are called Gal'tsov–Kerner glueballs.

The other simple possibility, $w(r) = 0$, $a(r) \neq 0$, is also nontrivial. The equations of motion in this case reduce to

$$\left(\frac{a'}{r^2\mathcal{R}}\right)' = -\left(\frac{r^2 a'}{\mathcal{R}}\right)' \;,\; \mathcal{R} = \sqrt{\frac{(1+r^4)(1-a'^2)}{r^4}} \;. \tag{10}$$

This equation can be solved explicitly and its solution is a two-parameter family

$$a(r;\ C,\ \alpha) = C \pm \int \sqrt{\frac{\alpha - 1}{\alpha + r^4}}\, dr \;, \tag{11}$$

where C and α are the integration constants and $\alpha > 1$. The explicit form of the solution is given in terms of the elliptic integral. In accordance with the conditions of finiteness of energy and invariance of the equations under the change $a(r) \to -a(r)$, we will take $C = 0$ and the $+$ sign in front of the square root. The function $a(r)$ for different values of α is shown in the Figure 1. The limiting value of the parameter, $\alpha = 1$, gives $a(r) =$ const, a configuration which is gauge equivalent to the embedded monopole $w(r) = 0$, $a(r) = 0$. The energy of the solution (11) is

$$M(\alpha) = \frac{\pi^{3/2}}{\Gamma(3/4)^2}\,\frac{1}{2\alpha^{1/4}}\left(1 - \frac{\alpha}{3}\right) \;. \tag{12}$$

The energy is unbounded below and its maximum is M_e at $\alpha = 1$. We observe that the existence of the electric field decreases the total energy.

We will call the solution (11) dyon [4], as in the asymptotic region $r \to \infty$ the behavior of the electric and magnetic fields is given by

$$E_i^a \sim \sqrt{\alpha - 1}\,\frac{x_i x_a}{r^4} \;,\quad B_i^a \sim -\frac{x_i x_a}{r^4} \;, \tag{13}$$

and describes the field strenghts of point-like sources. The "electric charge" of the source is proportional to $\sqrt{\alpha - 1}$, while the "magnetic charge" is 1.

3. Conclusions

The condition of finiteness of energy in the general case of spherically symmetric NBI configurations restricts the possible behavior of the functions $w(r)$ and $a(r)$. Along with the cases discussed above, there is one interesting solution which behaves as 't Hooft-Polyakov monopole (i.e.

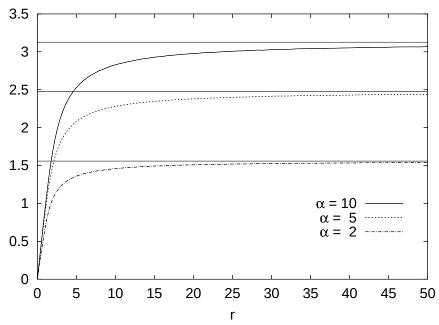

Figure 1: Dyon solutions

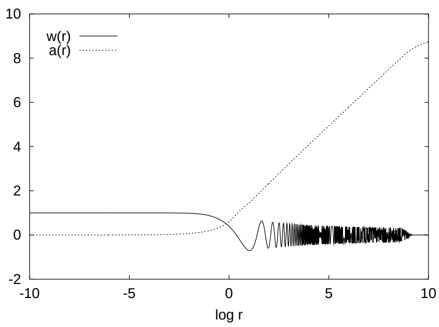

Figure 2: Would-be NBI monopole

$a(r) \to$ const and $w(r) \to 0$ for $r \to \infty$). It can be obtained by numerical integration starting from $r = 0$ for some values of initial parameters denoted as w_2 and a_1. The outcome of the integration for $w_2 = -10$ and $a_1 = 0.5$ with the integration step $h = 10^{-5}$ is shown in the Figure 2. However, this solution is numerically unstable: if we keep the same values for the parameters w_2 and a_1, but decrease the integration step h, we see that the oscillations of $w(r)$ increase to the larger region of r and that the asymptotic value of $a(r)$ at infinity increases. We conclude that the solutions of this type are probably nonanalytic. The analysis of the energy confirms this conclusion, too: the values of energy differ for orders of magnitude for different integration steps. We see that in the NBI case, as in the pure Yang–Mills theory, $w(r) = 0$ and $w(r) = 1$ are separated by infinite energy barrier and it is impossible to find the solution of finite energy which interpolates between them.

Further numerical analysis of the other allowed asymptotics strongly indicates that, besides Gal'tsov–Kerner glueballs and analytic dyons, there are no finite energy solutions.

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